COINCIDENCE INVARIANTS AND HIGHER REIDEMEISTER TRACES

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Abstract. Coincidence invariants generalize either the classical Lefschetz number, and apply in extremely limited examples, or are bordism invariants that apply generally but are difficult to compute.

In this paper we compare these alternatives using traces in bicategories. This approach suggests further generalizations of the classical approach and significantly simplifies the proofs of previously known comparison results.

1. Introduction

A *coincidence point* for a pair of maps $f, g\colon M \to N$ is a point x of M such that $f(x) = g(x)$. Coincidence points are an easy generalization of fixed points and there is a corresponding generalization of the Lefschetz fixed point theorem.

Theorem 1.1. [16] *Suppose M and N are closed, smooth, orientable manifolds of the same dimension and $f, g\colon M \to N$ are continuous maps. If f and g have no coincidence points then*

$$L(f,g) \coloneqq \sum_i (-1)^i \mathrm{tr} \left(\begin{tikzcd} H_i(M;\mathbb{Q}) \arrow[r, "f_*"] & H_i(N;\mathbb{Q}) & H_i(M;\mathbb{Q}) \\ & H^{\dim(N)-i}(N;\mathbb{Q}) \arrow[u, "-\cap[N]"] \arrow[r, "g^*"] & H^{\dim(N)-i}(M;\mathbb{Q}) \arrow[u, "-\cap[M]"] \end{tikzcd} \right) = 0.$$

The vertical maps are the Poincaré duality isomorphism. There is also a converse to this result where $L(f,g)$ is replaced by a generalization of the Nielsen number, [25].

The hypotheses of this theorem are extremely restrictive, and it is clear that they cannot be relaxed without significant changes to the invariant. One possibility is the bordism approach used in [12, 13, 14, 15].

Theorem 1.2. [12, Theorem A] *Maps $f, g\colon M \to N$ determine a vector bundle ξ over the twisted loop space*

$$\Lambda^{f,g}N \coloneqq \{(x,\gamma) \in M \times N^I | f(x) = \gamma(0), g(x) = \gamma(1)\}$$

and an element $\chi(f,g)$ in the framed bordism group $\Omega_{\dim(M)-\dim(N)}(\Lambda^{f,g}N, \xi)$. If f and g are coincidence free, $\chi(f,g)$ is trivial.

If $\dim(M) + 3 \leq 2\dim(N)$, f and g are homotopic to coincidence free maps if and only if $\chi(f,g)$ is trivial.

Date: September 17, 2012.

In this paper we will show the invariants in these theorems are closely related. We will use the symmetric monoidal and bicategorical traces of [6, 19, 22] to extend the classical fixed point invariant comparison results to coincidences and produce topological invariants that are very similar to the algebraic invariant in Theorem 1.1.

There is a change in perspective that is an important part of these comparisons. We usually think of the classical Lefschetz number as an integer and the Reidemeister trace as an element of the free abelian group on the set of fixed point classes, but these invariants also admit descriptions as elements of stable homotopy groups. In the case of fixed points, the relevant stable homotopy groups happen to be isomorphic to the simpler groups used in the classical approach. For some coincidences these comparisons fail and the classical descriptions of the fixed point invariants do not generalize easily. However, the homotopy theoretic descriptions generalize easily in all cases.

These homotopy theoretic descriptions can be difficult to define and to compare to previously defined invariants. Surprisingly, the bicategory of parameterized spectra provides a way to avoid many of these challenges. We make significant use of it here. It also simplifies some of the homotopy theory in [19] and is extremely useful in [21].

Remark 1.3. A *parameterized space* over a space B is a space E along with maps $\sigma\colon B \to E$, the *section*, and $p\colon E \to B$, the *projection*, such that $p \circ \sigma$ is the identity map of B. A map of parameterized spaces preserves both section and projection.

There is an *external smash product* for parameterized spaces. If E is a parameterized space over B and E' is a parameterized space over B' the external smash product is a parameterized space over $B \times B'$ and the fiber over (b, b') is the smash product of the fiber of E over b and the fiber of E' over b'. If E and E' are both parameterized spaces over B, there is also an internal product. This is formed by taking the external smash product, pulling back along the diagonal map of B and then quotienting out the section. This produces a based topological space.

To notate these products we follow [18]. If E is a parameterized space over $A \times B$ we regard it as a space over A on the left and a space over B on the right. If E' is a parameterized spaced over $B \times C$, $E \odot E'$ is the space over $A \times C$ given by first taking the external smash product of E and E' and then internalizing B. If E is a parameterized space over B, we regard it as a space over B on the right. If we want to regard it as a space over B on the left we write $\widehat{E}$. Then if E' is a space over B, $E \odot \widehat{E'}$ is the internal product of E and E' and $\widehat{E'} \odot E$ is the external smash product of E' and E.

For a manifold M, $S^0_M \coloneqq M \amalg M$ where the projection is the identity map on each factor and the section in the inclusion into the first factor. If M embeds in a space Q, $S^{\nu_{M \subset Q}}$ is the fiberwise one point compactification of the normal bundle of this embedding. This is a parameterized space over M, the projection is induced by the projection map for the bundle and the section is the section at infinity. If $f\colon M \to N$ is a continuous map

$$_fS \coloneqq \{(x, \gamma) \in M \times N^I | f(x) = \gamma(0)\} \amalg (M \times N)$$

and is regarded as a space over $M \times N$ using the map $(x, \gamma) \mapsto (x, \gamma(1))$ and the identity map. The spaces S_f and $_fS_g$ are similar.

For more information about parameterized homotopy theory see [18].

2. Coincidence Indices

The starting point for the approach to coincidence points in [7, 8, 9, 12, 13, 14, 15, 23, 24] is the observation that the coincidence points of maps $f, g\colon M \to N$ are the intersection of the diagonal in N with the image of the product

$$f \times g\colon M \to N \times N.$$

So instead of focusing specifically on coincidence points, we will first consider intersections of submanifolds more generally. Let Q be a submanifold of a manifold P and $f\colon M \to P$ be a continuous map. One simple way to homotopically measure the intersection of Q and the image of f is to consider the composite of f with the Thom collapse for the normal bundle of Q in P

$$M \xrightarrow{f} P \longrightarrow T\nu_{Q\subset P}\ . \tag{2.1}$$

If Q and the image of f are disjoint, this composite is homotopically trivial. In fact, it is homotopically trivial if f is homotopic to a map g whose image is disjoint from Q. This composite will give the comparison between the invariants in Theorems 1.1 and 1.2.

First we need another description of the stable homotopy class of this map. Let $\{-,-\}$ denote stable homotopy classes of maps.

Theorem 2.2. [1, 6, 17] *If M is a closed smooth manifold that embeds in $\mathbb{R}^{m+p}$, there is an isomorphism*

$$\{X \wedge M_+, Y\} \cong \{X \wedge S^{m+p}, Y \wedge T\nu_M\}$$

for based spaces X and Y.

Here ν_M denotes the normal bundle for the embedding of M in $\mathbb{R}^{m+p}$ and $T\nu_M$ is the Thom space for that bundle. The theorem above is *Spanier-Whitehead duality* and the isomorphism is induced by the composite

$$\eta\colon S^{m+p} \longrightarrow T\nu_M \longrightarrow M_+ \wedge T\nu_M$$

of the Thom collapse of the embedding of M in $\mathbb{R}^{m+p}$ and the Thom diagonal.

Definition 2.3. The *intersection index* of f and Q is the Spanier-Whitehead dual of the composite in 2.1.

The intersection index is a refinement of the classical Lefschetz number for fixed points. To recover more familiar and computationally tractable invariants for coincidence points we need to simplify the intersection index. We will simplify by composing with another map. A first example to consider is when both M and N have trivial normal bundles. In this case, there is a stable map

$$\vartheta\colon T\nu_{\triangle\subset N\times N} \wedge T\nu_M \to S^{n+p} \tag{2.4}$$

given by trivializing the bundles and then projecting. Composing this map with the intersection index defines the *coincidence index* of f and g relative to ϑ, $i_\vartheta(f,g)$. Note that the Spanier-Whitehead dual of the map in 2.4 is a stable map

$$\theta\colon S^m \wedge T\nu_{\triangle\subset N\times N} \to S^n \wedge M_+$$

and so the coincidence index is the symmetric monoidal trace [6, 17, 20] of the composite

$$S^m \wedge M_+ \xrightarrow{\mathrm{id}\wedge(f\times g)} S^m \wedge T\nu_{\triangle\subset N\times N} \xrightarrow{\theta} S^n \wedge M_+\ .$$

Requiring trivial normal bundles is a much more restrictive hypothesis than necessary. A simple alternative is to assume the manifolds are orientable as in Theorem 1.1. Let k_* be a homology theory and suppose M and N are k_*-orientable. If k is the associated spectrum, there are Thom isomorphisms, [18, 20.5.8],

$$k \wedge T\nu_M \cong k \wedge \Sigma^p M_+ \quad \text{and} \quad k \wedge T\nu_{\triangle \subset N\times N} \cong k \wedge \Sigma^n N_+.$$

The projection map for N and the Thom collapse for an embedding of M in $\mathbb{R}^{m+p}$ define a stable map

$$\begin{tikzcd}
\theta\colon k \wedge S^{m+p} \wedge T\nu_{\triangle\subset N\times N} \arrow[d, "\sim"] & & k \wedge S^{p+n} \wedge M_+ \, . \\
k \wedge S^{m+p+n} \wedge N_+ \arrow[r] & k \wedge S^{m+p+n} \arrow[r] & k \wedge S^n \wedge T\nu_M \arrow[u, "\sim"']
\end{tikzcd}$$

and the k_*-*index* of f and g is the symmetric monoidal trace of the composite

$$k \wedge S^m \wedge M_+ \xrightarrow{\mathrm{id}\wedge\mathrm{id}\wedge(f\times g)} k \wedge S^m \wedge T\nu_{\triangle\subset N\times N} \xrightarrow{\theta} k \wedge S^n \wedge M_+.$$

This uses the generalization of the symmetric monoidal trace defined in [20] and is a stable map $k \wedge S^m \to k \wedge S^n$. It induces a map

$$\tilde{k}_*(S^m) \to \tilde{k}_*(S^n).$$

When there is no ambiguity, this map will also be called the k_*-*index* of f and g.

The Thom isomorphisms also define isomorphisms

$$\tilde{k}_*(T\nu_M) \cong \tilde{k}_*(\Sigma^p M_+) \quad \text{and} \quad \tilde{k}_*(T\nu_{\triangle\subset N\times N}) \cong \tilde{k}_*(\Sigma^n N_+).$$

Following standard terminology, k_*-*Lefschetz number* of f and g is the trace of the composite

(2.5)
$$\begin{tikzcd}
\tilde{k}_*(M_+) \arrow[r, "f_*"] & \tilde{k}_*(N_+) \arrow[d, "\cong"] & \tilde{k}_{*+q-p}(M_+) \\
& \tilde{k}_{*+q}(T\nu_N) \arrow[r, "(Dg)_*"] & \tilde{k}_{*+q}(T\nu_M) \arrow[u, "\cong"']
\end{tikzcd}$$

(Note that Poincaré duality is the composite of Spanier-Whitehead duality and the Thom isomorphism so this agrees with the invariant in Theorem 1.1.)

Theorem 2.6. *If k_* has a Künneth isomorphism the k_*-Lefschetz number and k_*-index coincide.*

Proof. A diagram chase shows the trace of the composite

$$\tilde{k}_*(S^m \wedge M_+) \xrightarrow{k_*(f\times g)} \tilde{k}_*(T\nu_{\triangle\subset N\times N}) \xrightarrow{k_*(\theta)} \tilde{k}_*(S^n \wedge M_+)$$

is the trace of of the composite in 2.5.

If M_+ is dualizable and k_* satisfies a Künneth isomorphism, then $\tilde{k}_*(M_+)$ is also dualizable. The result then follows from the independence of the symmetric monoidal trace from the choice of dual pair, [6, 20]. $\square$

There is no dimension condition in the theorem, but if k_* is rational homology and the dimensions of M and N are not the same both the index and the Lefschetz number are trivial. This is the first of many examples that show the limitations of the purely algebraic techniques.

This approach is compatible with the local coincidence index, [25, 26]. After replacing f and g by homotopic maps we may assume the coincidence points are a submanifold of M. Let X be a component of coincidence points and U be a tubular neighborhood of X containing no other coincidence points. The composite below generalizes the coincidence index.

$$\begin{tikzcd} M \arrow[r] & T\nu_{X\subset M} & T\nu_{\triangle\subset N\times N} \\ & (U, U\setminus X) \arrow[u, "\sim"] \arrow[r, "f\times g"] & (N\times N, N\times N\setminus\triangle) \arrow[u, "\sim"] \end{tikzcd}$$

The first map is the Thom collapse for the inclusion of X in M. The mapping cone on the inclusion of $U\setminus X$ into U is denoted $(U, U\setminus X)$ and the vertical maps are comparisons of homotopy cofibers and quotients and are equivalences in this case. The dual of the composite is a map

$$S^{m+p}\to T\nu_{\triangle\subset N\times N}\wedge T\nu_M$$

and if M and N are k_*-orientable the composite with θ above defines the *k_*-index* of f and g at X. (Note that this does not depend on the choice of U.)

If the dimensions of M and N are the same we may assume that the coincidence points are isolated and the discussion above defines a homomorphism

$$k_*(S^m)\to k_*(S^m)$$

for each open set U of M.

Proposition 2.7. *If the dimensions of M and N are the equal the $H_*(-;\mathbb{Q})$-index is the invariant defined in* [26].

Proof. This follows from the axiomatization in [26, Theorem 5]. The normalization axiom follows from Theorem 2.6 above. The additivity follows from a factoring of the index and the homotopy axiom is automatic. □

The orientability hypothesis above is much more restrictive than necessary. All that is needed to produce a theorem like Theorem 1.1 is a stable map

$$\theta\colon T\nu_{Q\subset P}\wedge K\to L\wedge M_+$$

for spaces K and L. (Using Spanier-Whitehead duality this is equivalent to a map $T\nu_{Q\subset P}\wedge M_+\wedge K\to L$.) The *intersection index* of $f\colon M\to P$ and Q relative to θ is the symmetric monoidal trace of the composite

$$K\wedge M_+\wedge T\nu_M\xrightarrow{\mathrm{id}\wedge f\wedge \mathrm{id}} K\wedge T\nu_{Q\subset P}\wedge T\nu_M\xrightarrow{\theta\wedge\mathrm{id}} L\wedge M_+\wedge T\nu_M$$

and it is clear that the index of f and Q is trivial if the image of f and Q do not intersect. There is also a corresponding *Lefschetz number*. It is defined to be the symmetric monoidal trace [20] of the composite

$$H_*(K)\otimes H_*(M)\xrightarrow{\mathrm{id}\otimes f_*} H_*(K)\otimes H_*(T\nu_{Q\subset P})\xrightarrow{\theta_*} H_*(L)\otimes H_*(M)$$

Functoriality of the symmetric monoidal trace implies the following result.

Theorem 2.8. *The map induced on homology by the intersections index of f and Q relative to θ is the same as the Lefschetz number relative to $H_*(\theta)$.*

This is consistent with the invariants defined in [10, 23, 24] and provides alternative proofs for the main comparison results of those papers. In [10, Theorem 3.1], the choices of homology and cohomology classes determine a choice of the map θ as above. The situation is similar in [23, Theorem 2.3] and [24, Theorem 6.1]. The primary difference is that the dualizability assumption is placed on N rather than M.

3. The fiberwise Thom collapse

The invariants considered in the previous section generalize the classical Lefschetz number and fixed point index. For generalizations of the Nielsen number and Reidemeister trace we need to replace the classical Thom collapse by a fiberwise Thom collapse. There are several descriptions of the fiberwise Thom collapse, we will start with the version used in [12].

Returning to the example of a manifold P with a submanifold Q and a continuous map $f\colon M \to P$, the *fiberwise unreduced mapping cylinder* $C_P(P, P\setminus Q)$ of the inclusion $P\setminus Q \to P$ is the homotopy pushout of the inclusions

$$\begin{tikzcd} P\setminus Q \arrow[r] \arrow[d] & P \\ P. \end{tikzcd}$$

There is an obvious fiberwise inclusion $S^0_P \coloneqq P \amalg P \to C_P(P, P\setminus Q)$.

A map $f\colon M \to P$ defines a commutative diagram

$$\begin{tikzcd} S^0_M \arrow[r, "f"] \arrow[d] & S^0_P \arrow[d] \\ M \arrow[r, "f"] & P \end{tikzcd}$$

and so defines a fiberwise map $S^0_M \to S^0_P \odot S_f$. Composing with the inclusion above, we have a fiberwise map

$$S^0_M \to C_P(P, P\setminus Q)\odot S_f \tag{3.1}$$

generalizing the composite in 2.1.

Theorem 3.2. [12, Theorem 3.4] *If $\dim(M) + 2\dim(Q) + 3 \leq 2\dim(P)$, the fiberwise stable homotopy class of the composite in 3.1 is trivial if and only if there is a map $g\colon M \to P$, homotopic to f, such that the image of g and Q are disjoint.*

In the case of coincidences of $f, g\colon M^m \to N^n$ this result becomes the following statement.

Corollary 3.3. *If $m + 3 \leq 2n$, the stable homotopy class of the composite*

$$S^0_M \longrightarrow S^0_{N\times N}\odot S_{f\times g} \longrightarrow C_{N\times N}(N\times N, N\times N\setminus \triangle)\odot S_{f\times g}$$

is trivial if and only if there are maps $f', g'\colon M \to N$ homotopic to f and g respectively, such that f' and g' have no coincidences.

This description is difficult to compare directly with classical fixed and coincidence point invariants. To simplify these comparisons we will modify results in [12] to use fiberwise rather than equivariant homotopy and follow the notation of [18].

Proposition 3.4. [12, 5.1] *There is a fiberwise equivalence*

$$C_P(P, P\setminus Q)\to S^{\nu_{Q\subset P}}\odot {}_{i_Q}S.$$

We postpone the proof of this proposition until the end of the section, but it is not difficult to describe the composite

$$\psi\colon S^0_P\to C_P(P,P\setminus Q)\to S^{\nu_{Q\subset P}}\odot {}_{i_Q}S.$$

Let ϕ be a diffeomorphism from a tubular neighborhood T of Q in P to the normal bundle $\nu_{Q\subset P}$ of Q in P. Define a map

$$t\colon \nu_{Q\subset P}\to P^I \tag{3.5}$$

by $t(v)(s)=\phi^{-1}(sv)$. If σ is the section of $S^{\nu_{Q\subset P}}\odot {}_{i_Q}S$, ψ is defined by

$$\psi(x)=\begin{cases}(t(x),\phi(x)) & x\in T\\ \sigma(x) & x\notin T.\end{cases}$$

This is the homotopy Pontryagin-Thom collapse in [3, §6] and [4, II.12]. Collapsing the sections and forgetting these were spaces over P, we obtain a map equivalent to the classical Thom collapse.

As with indices, it is useful to replace this map by its dual. The relevant notion of duality here is Costenoble-Waner duality.

Theorem 3.6. [18, 18.6.1]*If M is a closed smooth manifold that embeds in $\mathbb{R}^{m+p}$, there is a natural isomorphism*

$$\{X\odot S^0_M, Y\}_{A\times M}\cong\{X\wedge S^{m+p}, Y\odot\widehat{S^{\nu_M}}\}_{A\times *}$$

for all parameterized spaces X over $A\times *$ and Y over $A\times M$.

This isomorphism is induced by the classical Thom collapse map for an embedding $M\to\mathbb{R}^{m+p}$,

$$S^{m+p}\xrightarrow{\eta} T\nu_M\cong S^0_M\odot\widehat{S^{\nu_M}}$$

Definition 3.7. The *intersection Reidemeister trace* of f and Q is the Costenoble-Waner dual of the composite in 3.1.

In the remaining sections we consider special cases of this invariant for fixed points and coincidences points. We will show how to recover and then extend classical invariants using this approach.

Proof of Proposition 3.4. The disk bundle and sphere bundle of $\nu_{Q\subset P}$ can be regarded as spaces over P by inclusion and over Q using the bundle structure. The first will be denoted by $\nu_{Q\subset P,i}$, the second $\nu_{Q\subset P}$.

The equivalence is the composite,

$$\begin{tikzcd}
C_P(P,P\setminus Q) & C_Q(D(\nu_{Q\subset P}),S(\nu_{Q\subset P}))\odot {}_{i_Q}S \arrow[r, "\sim"] & S^{\nu_{Q\subset P}}\odot {}_{i_Q}S\\
C_P(D(\nu_{Q\subset P,i}),S(\nu_{Q\subset P,i})) \arrow[u, "\sim"] \arrow[r, "\sim"] & C_P(i_{Q!}D(\nu_{Q\subset P}), i_{Q!}S(\nu_{Q\subset P})) \arrow[u, "\sim"'] &
\end{tikzcd}$$

The last two equivalences are described in [18, 18.4.4, 18.4.5].

For the first equivalence, note $C_P(P, P \setminus Q)$ and $C_P(D(\nu_{Q\subset P,i}), S(\nu_{Q\subset P,i}))$ are homotopy pushouts. Since the square

$$\begin{tikzcd}
S(\nu_{Q\subset P,i}) \arrow[r] \arrow[d] & P \setminus Q \arrow[d] \\
D(\nu_{Q\subset P,i}) \arrow[r] & P
\end{tikzcd}$$

is also a homotopy pushout, the inclusion of $C_P(D(\nu_{Q\subset P,i}), S(\nu_{Q\subset P,i}))$ in $C_P(P, P \setminus Q)$ is an equivalence.

For the second equivalence recall $i_{Q!}D(\nu_{Q\subset P})$ is the homotopy pull back of the bottom right portion in the diagram below.

$$\begin{tikzcd}
D(\nu_{Q\subset P,i}) \arrow[dr, dotted] \arrow[drrr, hook, bend left] \arrow[ddr, equal] & & & \\
& i_{Q!}D(\nu_{Q\subset P}) \arrow[rr] \arrow[d] & & P \arrow[d, equal] \\
& D(\nu_{Q\subset P}) \arrow[r, "\pi"] & Q \arrow[r, hook] & P
\end{tikzcd}$$

The outside of the diagram commutes up to homotopy and so induces a fiberwise map from $D(\nu_{Q\subset P,i})$. This induced map is a fiberwise homotopy equivalence.

The case for the sphere bundle is similar. □

4. Fixed points

The intersection Reidemeister trace agrees with the classical Reidemeister trace for fixed points. To see this, we will first compare the intersection Reidemeister trace with a trace in the bicategory of parameterized spectra.

Theorem 4.1. *There is a map*

$$\epsilon\colon S^{\nu_{\triangle\subset M\times M}} \odot {}_{i_\triangle}S_{\Gamma_f} \odot \widehat{S^{\nu_M}} \to (\Lambda^f M)_+ \wedge S^{m+p}.$$

and the image of the intersection Reidemeister trace under this map is the bicategorical trace of

$$f\colon S^0_M \to S^0_M \odot S_f.$$

In this section this theorem primarily serves as a transition between the intersection Reidemeister trace and more classical fixed point invariants, however this description of the Reidemeister trace as a trace in the bicategory of spectra seems likely to have significant advantages. See, for example, [21].

Proof. We start by recalling the evaluation map for the dual pair (S^0_M, S^{ν_M}) in [18, 18.8]. For a (small enough) tubular neighborhood U of the diagonal in $M \times M$ there is a map $s\colon U \to M^I$ such that $s(x,y)(0) = x$ and $s(x,y)(1) = y$. Taking the derivative of s at 0 defines a map

$$\tau\colon U \to \tau_M.$$

The evaluation takes a normal vector v based at m and a point m' in M to the pair $(s(m,m'), v + \tau(m,m'))$. Similarly define a map

$$\epsilon\colon S^0_M \odot S_f \odot \widehat{S^{\nu_M}} \to (\Lambda^f M)_+ \wedge S^{m+p}$$

by $(v,\gamma,m)\mapsto(\gamma s(x,m),v+\tau(x,m))$ where v is a normal vector based at x. This is the shadow, [19], applied to the bicategorical evaluation of S^{ν_M} and S^0_M.

The neighborhood U can also be used to define the map in 3.5 giving

$$t_1\times t_2\colon U\to (M\times M)^I\cong M^I\times M^I$$

satisfying $(t_1\times t_2)(x,y)(0)=(x,y)$ and $t_1(x,y)(1)=t_2(x,y)(1)$. Note that U can be chosen so that s is homotopic to $t_2^{-1}t_1$.

A point in $S^{\nu_{\triangle\subset M\times M}}\odot{}_{i_\triangle}S_{\Gamma_f}\odot\widehat{S^{\nu_M}}$ outside of the section is a triple $(v,(\gamma_1,\gamma_2),w)$ where v is a normal vector to M based at a point x, w is a normal vector to the diagonal in $M\times M$ based at a point (y,y) and γ_1 and γ_2 are paths in M satisfying

$$\gamma_1(0)=f(x),\gamma_2(0)=x,\text{ and }\gamma_1(1)=y=\gamma_2(1).$$

Define a map

$$S^{\nu_{\triangle\subset M\times M}}\odot{}_{i_\triangle}S_{\Gamma_f}\odot\widehat{S^{\nu_M}}\longrightarrow(\Lambda^fM)_+\wedge S^{m+p}$$

by $(v,(\gamma_1,\gamma_2),w)\mapsto(\gamma_2^{-1}\gamma_1,v+(\gamma_1t_1(w))^*\tau(w))$ where $(\gamma_1t_1(w))^*\tau(w)$ is the translation of $\tau(w)$ along the path $\gamma_1t_1(w)$.

Then the diagram below commutes up to homotopy.

$$\begin{tikzcd}
S^0_M\odot\widehat{S^{\nu_M}} \arrow[r, "T(\Gamma_f)\wedge\mathrm{id}"] \arrow[dr, "f\wedge\mathrm{id}"'] & S^{\nu_{\triangle\subset M\times M}}\odot{}_{i_\triangle}S_{\Gamma_f}\odot\widehat{S^{\nu_M}} \arrow[r, "\epsilon"] & (\Lambda^fM)_+\wedge S^{m+p} \\
& S^0_M\odot S_f\odot\widehat{S^{\nu_M}} \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

□

To complete the comparison to the classical Reidemeister trace we will replace fiberwise homotopy theory with equivariant homotopy theory. This is similar to the technique used in [12].

Recall that the set of components of Λ^fM is isomorphic to the set of f_*-semiconjugacy classes of π_1M.

Corollary 4.2. *The image of the intersection Reidemeister trace under the isomorphism*

$$\pi_0^s(\Lambda^fM_+)\cong\pi_0^s(\langle\!\langle\pi_1(M)_{f_*}\rangle\!\rangle)$$

is the classical Reidemeister trace of f.

Proof. Let $\tilde M$ be the universal cover of M and let $p\colon\tilde M\to M$ be the usual quotient map. Then $\tilde M_{+M}:=\tilde M\amalg M$ is a parameterized space over M with projection $p\amalg\mathrm{id}$. Let $T_{\tilde M}\nu_M$ be $\tilde M_{+M}\odot\widehat{S^{\nu_M}}$.

If $\wedge_{\pi_1M}$ is the usual smash product followed by the quotient by the diagonal action there is a map

$$S^0_M\odot\widehat{S^{\nu_M}}\to\tilde M_+\wedge_{\pi_1M}T_{\tilde M}\nu_M$$

which takes (m,v) to $(\tilde m,\tilde m,v)$ where $\tilde m$ is any point in $\tilde M$ such that $p(\tilde m)=m$. There is a similar map

$$S^0_M\odot S_f\odot\widehat{S^{\nu_M}}\to\tilde M_+\wedge_{\pi_1M}(\pi_1M_{f_*})\wedge_{\pi_1M}T_{\tilde M}\nu_M$$

where $(\pi_1M)_{f_*}$ is π_1M as a left π_1M-set and has the usual right action twisted by f_*.

Then the diagram below commutes and compares the traces.

$$\begin{tikzcd}
S^{m+p} \arrow[r, "\eta"] \arrow[dr, "\eta"'] & S^0_M \odot \widehat{S^{\nu_M}} \arrow[r, "f\wedge \mathrm{id}"] \arrow[d] & S^0_M \odot S_f \odot \widehat{S^{\nu_M}} \arrow[r, "\epsilon"] \arrow[d] & (\Lambda^f M)_+ \wedge S^{m+p} \arrow[d, "\pi"] \\
& \tilde{M}_+ \wedge_{\pi_1 M} T_{\tilde{M}}\nu_M \arrow[r, "\tilde{f}\wedge \mathrm{id}"] & \tilde{M}_+ \wedge_{\pi_1 M} (\pi_1 M_{f_*}) \wedge_{\pi_1 M} T_{\tilde{M}}\nu_M \arrow[r, "\epsilon"] & S^{m+p} \wedge \langle\!\langle (\pi_1 M)_f \rangle\!\rangle_+
\end{tikzcd}$$

See [19, 3.1.3] for the definitions of the maps in the bottom composite. In [19, Proposition 3.2.3], the classical Reidemeister trace was identified with this composite. □

Corollary 4.3. *The image of the intersection Reidemeister trace under the map*

$$\pi_0^s(\Lambda^f M_+) \to \pi_0^s(S^0).$$

induced by the map $\Lambda^f M \to *$ is the fixed point index.

Proof. This follows from an argument very similar to that in the corollary above. In this case the vertical maps are given by collapsing the sections of the fiberwise spaces.

$$\begin{tikzcd}
S^{m+p} \arrow[r, "\eta"] \arrow[dr, "\eta"'] & S^0_M \odot \widehat{S^{\nu_M}} \arrow[r, "f\odot \mathrm{id}"] \arrow[d] & S^0_M \odot S_f \odot \widehat{S^{\nu_M}} \arrow[r, "\epsilon"] \arrow[d] & (\Lambda^f M)_+ \wedge S^{m+p} \arrow[d, "\pi"] \\
& M_+ \wedge T\nu_M \arrow[r, "f\wedge \mathrm{id}"] & M_+ \wedge T\nu_M \arrow[r, "\epsilon"] & S^{m+p}
\end{tikzcd}$$

See [6] for the definitions of the maps in the bottom composite. □

Note that the invariants here are all defined using topological techniques, but they can be identified with algebraic invariants either using classical techniques [2, 11] or the approach in [19].

Remark 4.4. Coincidences of a map

$$f\colon B \times F \to F$$

and the projection $\pi_2\colon B \times F \to F$ are pairs $(b, x) \in B \times F$ where $f(b, x) = x$. The index, Lefschetz number, and Reidemeister trace for these coincidences are very similar to the classical fixed point invariants.

These coincidences are detected by the symmetric monoidal trace of f, which, as above, is called the *index.* The usual functoriality arguments [6, 20] show that the map induced on homology by the index of f agrees with the trace of

$$f_*\colon H_*(B) \otimes H_*(F) \to H_*(F).$$

(This is a slight modification of the traditional trace. See [20].) This index agrees with the fiberwise index in [5].

The intersection Reidemeister trace applies to these coincidences and is a fiberwise stable map

$$S^0_B \longrightarrow S^{\nu_{\triangle \subset F\times F}} \odot {}_{i_\triangle}S_{\pi_1 \times f} \odot \widehat{S^{\nu_F}}$$

Exactly as in the classical case, there is a map

$$S^{\nu_{\triangle \subset F\times F}} \odot {}_{i_\triangle}S_{\pi_1 \times f} \odot \widehat{S^{\nu_F}} \longrightarrow S^n \wedge (\Lambda^f F)_+$$

and the image of the Reidemeister trace of f is the bicategorical trace of the induced map

$$S^0_{B\times F}\to S^0_F\odot S_f.$$

This agrees with the fiberwise Reidemeister trace in [19].

5. Coincidence invariants for orientable manifolds

Invariants for coincidences is similar, but not identical. As above, we start with a comparison.

Proposition 5.1. *If M and N are k_*-orientable there is a map*

$$\vartheta\colon S^{\nu_{\triangle\subset N\times N}}\odot {}_{i_\triangle}S_{f\times g}\odot \widehat{S^{\nu_M}}\wedge k\to \Lambda^{f,g}N_+\wedge S^{p+n}\wedge k.$$

This map is similar to the map ϑ in 2.4, but because of the asymmetry in Costenoble-Waner duality it does not seem to admit a dual form like that used in Theorem 2.6. This suggests that there may be problems producing an algebraic Reidemeister trace for coincidences like that for fixed points in [11].

Proof. The Thom isomorphism, [18, 20.5.8], gives equivalences

$$S^{\nu_M}\wedge k\cong S^p_M\wedge k\quad\text{ and }\quad S^{\nu_{\triangle\subset N\times N}}\wedge k\cong S^n_N\wedge k.$$

These define a map

$$S^{\nu_{\triangle\subset N\times N}}\odot {}_{i_\triangle}S_{f\times g}\odot \widehat{S^{\nu_M}}\wedge k\to S^n_N\odot {}_{i_\triangle}S_{f\times g}\odot\widehat{S^p_M}\wedge k.$$

Away from the section, elements of ${}_{i_\triangle}S_{f\times g}$ are triples (m,γ_1,γ_2) such that

$$f(m)=\gamma_1(0), g(m)=\gamma_2(0)\text{ and }\gamma_1(1)=\gamma_2(1),$$

and so composition defines a map

$$S^n_N\odot {}_{i_\triangle}S_{f\times g}\odot\widehat{S^p_M}\to\Lambda^{f,g}N_+.$$ □

The *Reidemeister trace* of f and g is the composite

$$\begin{tikzcd}
S^{m+p}\wedge k \arrow[d, "\eta"] & & S^{p+n}\wedge\Lambda^{f,g}N_+\wedge k \\
S^0_M\odot\widehat{S^{\nu_M}}\wedge k \arrow[rr, "(f\times g)\odot\mathrm{id}\wedge\mathrm{id}"] & & S^{\nu_{\triangle\subset N\times N}}\odot {}_{i_\triangle}S_{f\times g}\odot\widehat{S^{\nu_M}}\wedge k \arrow[u, "\theta"']
\end{tikzcd}$$

Corollary 5.2. *The image of the Reidemeister trace under the map induced by the constant map $\Lambda^{f,g}N\to *$ is the index of f and g.*

Proof. The proof is very similar to the proof of Corollary 4.2. In the commutative diagram below the first two vertical maps are given by quotienting out the section.

$$\begin{tikzcd}
S^{m+p} \arrow[r, "\eta"] \arrow[dr, "\eta"'] & S^0_M\odot\widehat{S^{\nu_M}} \arrow[r, "(f\times g)\odot\mathrm{id}"] \arrow[d] & S^{\nu_{\triangle\subset N\times N}}\odot {}_{i_\triangle}S_{f\times g}\odot\widehat{S^{\nu_M}} \arrow[r, "\vartheta"] \arrow[d] & (\Lambda^{f,g}N)_+\wedge S^{p+n} \arrow[d, "\pi"] \\
& M_+\wedge T\nu_M \arrow[r, "(f\times g)\wedge\mathrm{id}"] & T\nu_{\triangle\subset N\times N}\wedge T\nu_M \arrow[r, "\vartheta"] & S^{p+n}
\end{tikzcd}$$

(The k's have been suppressed for readability.) □

There is also a local description of the Reidemeister trace. After replacing f and g by homotopic maps, we may assume that the set of coincidences points is a submanifold of M. If X is a component of this submanifold and U is a tubular neighborhood of X in M that contains no other coincidence points consider the composite

$$\begin{tikzcd} S_M^0 \arrow[r] & S_M(M, M\setminus X) & & S^{\nu_{\triangle\subset N\times N}} \odot {}_{i_\triangle}S_{f\times g} \\ & S_M(U, U\setminus X) \arrow[u] \arrow[rr, "f\times g"] & & S_{N\times N}(N\times N, N\times N\setminus \triangle)\odot S_{f\times g} \arrow[u] \end{tikzcd}$$

Here U and X are regarded as spaces over M by inclusion. The left vertical map is an equivalence since the diagram of inclusions

$$\begin{tikzcd} U\setminus X \arrow[r] \arrow[d] & M\setminus X \arrow[d] \\ U \arrow[r] & M \end{tikzcd}$$

is a homotopy pushout. The Spanier-Whitehead dual of the composite above is a map

$$S^{m+p} \longrightarrow S^{\nu_{\triangle\subset N\times N}} \odot {}_{i_\triangle}S_{f\times g} \odot \widehat{S^{\nu_M}}$$

If M and N are k_*-orientable, this defines a map

$$\tilde{k}_*(S^{m+p}) \to \tilde{k}_*(S^{p+n}\wedge(\Lambda^{f,g}N)_+),$$

the k_*-*index of* X, that is independent of U.

If the dimensions of M and N are the same, the $H_*(-;\mathbb{Q})$-index has an interpretation using indices and coincidence classes. We may assume that the coincidence points are isolated, so for each coincidence point x there an open set U containing x and containing no other fixed points. Then the argument above defines a homomorphism

$$\tilde{H}_*(S^0;\mathbb{Q}) \to \tilde{H}_*(\Lambda^{f,g}N_+;\mathbb{Q})$$

and so defines an element $c_x\gamma_x$ of $\mathbb{Q}\pi_0(\Lambda^{f,g}N)$. The path γ_x is the constant path at $f(x) = g(x)$ and c_x is the coincidence index of x.

Corollary 5.3. *If the dimensions of M and N are the same, the Reidemeister trace defined here is the same as the invariant defined in* [27].

Proof. This follows from the axiomatization of the local Reidemeister trace in [27, Theorem 4]. □

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